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DOUBLE GRAPHS

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Abstract : In this paper we study the elementary properties of double graphs, i.e. of graphs which are the direct product of a simple graph G with the graph obtained by the complete graph K_2 adding a loop to each vertex.

Keywords: Direct product; Lexicographic product; Cospectral graphs; Strongly regular graphs; Complexity; Graph homomorphisms; Retract; Chromatic index; Fibonacci cubes

1. INTRODUCTION

In [18] it was observed that the binary strings of length $n + 1$ without zigzags, i.e. without 010 and 101 as factors, can be reduced to the Fibonacci strings, i.e. binary strings without two consecutive 1's, of length n. The set of Fibonacci strings can be endowed with a graph structure saying that two strings are adjacent when they differ exactly in one position. The graphs obtained in this way are called Fibonacci cubes [12] and have been studied in several recent papers. We wondered if the set of all binary strings without zigzags could be endowed with some graph structure related in some way with Fibonacci cubes. One interesting such graph structure is the one induced by the graph structure of Fibonacci strings, that is the one obtained defining the adjacency saying that two binary strings without zigzags are adjacent if and only if the corresponding Fibonacci strings are adjacent as vertices of the Fibonacci cube. The resulting graph can be build up taking two distinct copies of the Fibonacci cube Γ_n and joining every vertex v in one component to every vertex w' in the other component corresponding to a vertex w adjacent to v in the first component. At this point it was straightforward to observe that this is a general construction which can be performed on every simple graph. We called double graphs all the graphs which can be obtained in such a way. Since the class of double graphs with this construction turned out to have several interesting properties, we decided to write this paper as an elementary introduction to such graphs that perhaps deserve to be better known.

2. Definitions

In this paper we will consider only finite simple graphs (i.e. without loops and multiple edges). As usual V(G) and E(G) denote the set of vertices and edges of G, respectively, and adj denote the adjacency relation of G. For all definitions not given here see [1,5,9,13,17].

The direct product of two graphs G and H is the graph G × H with V (G×H)=V(G) × V (H) and with adjacency defined by (v_1, w_1) adj (v_2, w_2) if and only if v_1 adj v_2 in G and w_1 adj w2 in H.

The total graph T_n, on n vertices is the graph associated to the total relation (where every vertex is adjacent to every vertex). It can be obtained from the complete graph K_n by adding a loop to every vertex. In [13] it is denoted by K_n^s.

We define the double of a simple graph G as the graph $\mathcal{D}$[G] = G × T_2. Since the direct product of a simple graph with any graph is always a simple graph, it follows that the double of a simple graph is still a simple graph.

In $\mathcal{D}$[G] we have (v, h) adj (w, k) if and only if v adj w in G. Then, if V (T_2) = {0, 1}, we have that $G_0 = \{(v,0) : v \in V(G)\}$ and $G_1 = \{(v,1) : v \in V(G)\}$ are two subgraphs of $\mathcal{D}$[G] both isomorphic to G such that $G_0 \cap G_1 = \phi$ and $G_0 \cup G_1$ is a spanning subgraph of $\mathcal{D}$[G]. Moreover we have an edge between (v. 0) and (w, 1) and similarly we have an edge between (v, 1) and (w, 0) whenever v adj w in G. We will call {G_0, G1} the canonical decomposition of [G]. See Fig. I for some examples.

From the above observations it follows that if G has n vertices and m edges then $\mathcal{D}$[G] has 2n vertices and 4m edges. In particular $\deg_{D[G]}(v, k) = 2\deg_G(v)$.

The lexicographic product (or composition) of two graphs G and H is the graph G o H with V (G) × V (H) as vertex set and with adjacency defined by (v_1, w_1) adj (v_2, w_2) if and only if v_1 = v_2 and w_1 adj w_2 in H or v_1 adj v_2 in G. The graph G o H can be obtained from G substituting to each vertex u of G a copy H_v of H and joining every vertex of H_v with every vertex of H_w whenever v and w are adjacent in G [13, p. 185].

Lemma 1. For any graph G we have G × T_n = G o N_n, where N_n is the graph on n vertices without edges.

Proof. For simplicity consider T_n and N_n on the same vertex set. Then the function $f : G \times T_n \rightarrow G \, o \, N_n$, defined by f (v, k) = (v, k) for every $(v,k) \in V(G \times T_n)$, is a graph isomorphism. Indeed, since N_n has no edges, we have that (v. h) adj (w, k) in G o N_n if and only if v adj w in G.

From Lemma 1 it immediately follows that:

Proposition 2. For any graph G on n vertices, $\mathcal{D}$[G] = G o N_2 and $\mathcal{D}$[G] is n-partite (Fig. 2).

We will write $\mathcal{D}^2$[G] for the double of the double of G. More generally we will have the graphs $D^k[G] = G \times T_{2^k} = G \, o \, N_{2^k}$, for every $k \in \mathbb{N}$.

The given definition of double graph can be generalized considering the operator $\mathcal{D}_k$ defined by $D_k[G] = G \times T_k$ for every simple graph G. For Lemma 1 it is also $\mathcal{D}_k$[G] = G o N_k for every simple graph G.

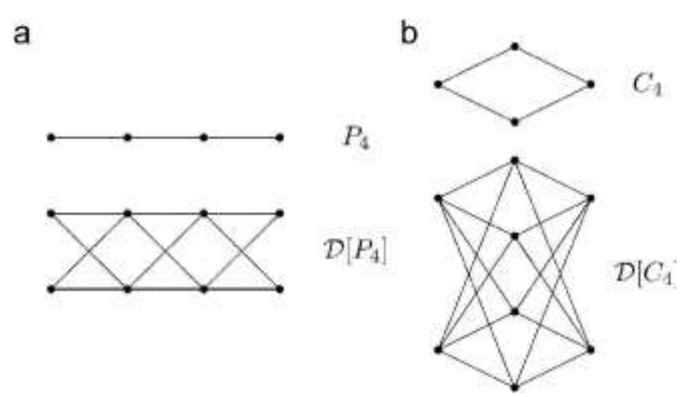

Fig. 1. (a) A path and its double, (b) a cycle and its double.

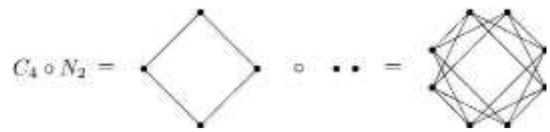

Fig. 2. The double of a 4-cycle drawn as lexicographic product.

Moreover the powers of $\mathcal{D}$ are instances of these generalized operators. Specifically $D^k[G] = D_{2^k}[G]$ for every simple graph G. Many of the properties proved in the sequel for $\mathcal{D}$ can be immediately extended to $\mathcal{D}_k$.

3. Basic properties of double graphs

In this section we will review some elementary properties of double graphs.

Proposition 3. The double $\mathcal{D}$[G] of a graph G on n vertices contains at least 2^n subgraphs isomorphic to G itself.

Proof. Let {G_0. G_1} be the canonical decomposition of $\mathcal{D}$[G]. Let S_0 be any subset of V (G_0) and let S_1 be the subset of V (G_1) corresponding to the complementary set of S_0. Then the graph induced by $S_0 \cup S_1$ is isomorphic to G.

Proposition 4. For any graph G, G is bipartite if and only if $\mathcal{D}$[G] is bipartite.

Proof. Let {G_0. G_1} be the canonical decomposition of $\mathcal{D}$[G]. If G is bipartite then also G_0 and G_1 are bipartite. Let{V, W} be a bipartition of G and (V_0. W_0), (V_1, W_1) be the corresponding bipartitions of G_0 and G_1, respectively. Every edge of $\mathcal{D}$[G] has one extreme in $V_0 \cup V_1$ and the other in $W_0 \cup W_1$, and hence also $\mathcal{D}$[G] is bipartite.

Conversely, if $\mathcal{D}$[G] is bipartite then it does not contain odd cycles. Hence also the subgraph $G_0 \cdot G$ does not contain odd cycles and then it is bipartite.

A vertex cut of a graph G is a subset S of V (G) such that G/S is disconnected. The connectivity k(G) of G is the smallest size of a vertex cut of G. A point of articulation (resp. bridge) is a vertex (resp. edge) whose removal augments the number of connected components. A block is a connected graph without articulation points.

Proposition 5. For any graph G $\neq K_1$ the following properties hold:

1. G is connected if and only if $\mathcal{D}$[G] is connected.
2. If G is connected then every pair of vertices of $\mathcal{D}$[G] belongs to a cycle.
3. Every edge of $\mathcal{D}$[G] belongs to a 4-cycle.
4. In a double graph there are neither bridges nor articulation points.
5. If G is connected then $\mathcal{D}$[G] is a block.
6. The connectivity of $\mathcal{D}$[G] is $k(\mathcal{D}[G]) = 2k(G)$.

Proof. Let (G_0, G_1) be the canonical decomposition of $\mathcal{D}$[G].

1. If G is connected also G_0 and G_1 are connected. Hence, we have only to prove that any vertex (v, 0) of G_0 is connected with any vertex of G_1. Let ν' be any vertex adjacent to ν. Then (v. (0) is adjacent to (v', 1). Since G_1 is connected there exists a path which connects (v', 1), and hence (v, 0), to any vertex of G_1. Conversely, if G is disconnected then also $\mathcal{D}$[G] is disconnected.

2. Let (v, 0) and (w, 0) be two distinct vertices in Go. Leto be a path connecting these two vertices and lety, be the corresponding path connecting the vertices (v, 1) and (w, 1) in G_1. Let (v', 1) be the vertex following (v. 1) on y_1. (w', 1) be the vertex preceding (w, 1) on y_1, and let γ'_1 be the sub-path of y_1 from (w', 1) to (v'. 1). Then $\gamma_0 \cup$ {(w, 0), (w', 1)} $\cup \gamma'_1 \cup$ {(v', 1), (v, 0)} is a cycle containing (v. 0) and (w, 0). A similar argument holds when we consider two distinct vertices in G_1 or two vertices (v, 0) and (w, 1) with $v \neq w$. Finally, in the case of two vertices (v. 0) and (v. 1), choosing any vertex v' adjacent to v, we have that (v, 0), (v', 1), (v, 1), (v'. 0), (1, 0) is a cycle containing both the vertices.
3. Every edge vw of G generates the 4-cycle (v. 0), (w, 0), (v, 1), (w, 1), (u, 0) in D[G] and every edge of D[G] belongs to one such cycle.
4. An edge vw of a connected graph H is a bridge if and only if no cycle of H contains both and w [17]. Here, without loss of generality, we can suppose G connected. Since every edge of D[G] belongs to a cycle it follows that D[G] has no bridges. Similarly, by property 2, G has no articulation points.
5. It follows from properties 1 and 4.
6. Let S be a vertex cut of D[G] with minimum size. The sets $S_0 = S \cap V(G_0)$ and $S_1 = S \cap V(G_1)$ are vertex cuts of G_0 and G_1, respectively. Then $|S_0|,|S_1| \geq k(G)$ and hence $k(D[G]) \geq 2k(G)$. Conversely, let S be a vertex cut of G and S_0 and S_1 be the corresponding sets in G_0 and G_1, respectively. Then $S_0 \cup S_1$ is a vertex cut of D[G] and hence $k(D[G]) \leq 2k(G)$.

A connected graph G is Eulerian if it has a closed trial containing all the edges of G. Eulerian graphs are characterized as the even connected graphs, where an even graph is a graph in which every vertex has even degree. A graph G is Hamiltonian if it has a spanning cycle.

Proposition 6. For any graph $G \neq K_1$ the following traversability properties hold:

1. If G is connected then D[G] is Eulerian.
2. If G is Hamiltonian then also D[G] is Hamiltonian.

Proof. 1. The double of a connected graph is connected and double graphs are always even.

2. Let {G_0, G_1) be the canonical decomposition of D[G]. Let y be a spanning cycle of G, vw be an edge of y and y' be the path obtained from y by removing the edge vw. Let γ_i' be the corresponding path in G_i, for i = 0, 1. Then $\gamma_0' \cup \{(w,0),(v,1)\} \cup \gamma_1' \cup \{(w,1),(v,0)\}$ is a spanning cycle of D[G].

Proposition 7. For any graph G_1 and G_2 the following properties hold:

1. $D[G_1 \times G_2] = G \times D[G_2] = D[G_1] \times G_2$,
2. $D[G_1 \text{ o } G_2] = G_1 \text{o } D[G_2]$

Proof. These identities are consequence of the associative property of the direct product and of the lexicographical product, respectively.

From the definition of the double of a graph it follows immediately that:

Proposition 8. Let A be the adjacency matrix of G. Then the adjacency matrix of D[G] is

$$D[A] = \begin{bmatrix} A & A \\ A & A \end{bmatrix} = A \otimes \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

The rank r (G) of a graph G is the rank of its adjacency matrix. Then from the above proposition it follows that:

Proposition 9. For any graph G, $r(D[G]) = r(G)$.

In the sequel we will use the property that two graphs are isomorphic if and only if their adjacency matrices are similar by means of a permutation matrix.

Let G_1 and G_2 be two graphs. The sum $G_1 + G_2$ of G_1 and G_2 is the disjoint union of the two graphs. The complete sum $G_1 \boxplus G_2$ of G_1 and G_2 is the graph obtained from $G_1 + G_2$ by joining every vertex of G_1 to every vertex of G_2. A graph is decomposable if it can be expressed as sums and complete sums of isolated vertices [17, p. 183].

Proposition 10. For any graph G_1 and G_2 the following properties hold:

1. $D[G_1+G_2] = D[G_1] + D[G_2]$.
2. $D[G_1 \boxplus G_2] = D[G_1] \boxplus D[G_2]$.
3. the double of a decomposable graph is decomposable.

Proof. The first two properties can be proved simultaneously as follows. Let A_1 and A_2 be the adjacency matrices of G_1 and G_2, respectively. Then $\begin{bmatrix} A_1 & X \\ X & A_2 \end{bmatrix}$ is the adjacency matrix of $G_1 + G_2$ when X is the null matrix O and of $G_1 \boxplus G_2$ when X is the matrix J all of whose entries are l's. Then the adjacency matrix of the double is

$$\begin{bmatrix} A_1 & X & A_1 & X \\ X & A_2 & X & A_2 \\ A_1 & X & A_1 & X \\ X & A_2 & X & A_2 \end{bmatrix}$$

Interchanging first the second and the third column and then the second and the third row we obtain the matrix

$$\begin{bmatrix} A_1 & A_1 & X & X \\ A_1 & A_1 & X & X \\ X & X & A_2 & A_2 \\ X & X & A_2 & A_2 \end{bmatrix}$$

which is the adjacency matrix of $D[G_1] + [G_2]$ when $X = O$ and of $D[G_1] \boxplus [G_2]$ when $X = J$. These properties are also implied by the right-distributive laws of the lexicographic product [13, pp. 185, 186]. Finally the third property follows from the fact that D preserves sums and complete sums and $D[K_1] = N_2 = K_1+K_1$.

Examples. 1. If N_n is the graph on n vertices without edges, then $D[N_n] = N_{2n}$.

2. Let $K_{m,n}$ be a complete bipartite graph. Then $D[K_{m,n}] = D[N_m \boxplus N_n] = D[N_m] \boxplus [N_n] = N_{2m} \boxplus N_{2n} = K_{2m.2n}$. Similarly, if $K_{m1,\ldots,mn}$ is a complete n-partite graph we have $D[K_{m1,\ldots,mn}] = K_{2m1,\ldots,2mn}$. In particular, if $K_{m(n)}$ is the complete m-partite graph $K_{n,\ldots,n}$, then $D[K_{m(n)}]=K_{m(2n)}$. Since $K_n = K_{n(1)}$ it follows that the double of the complete graph K_n is the hyperoctahedral graph $H_n = K_{n(2)}$

3. For $n \geq 2$, let K_n^- be the graph obtained by the complete graph K_n deleting any edge. Then $K_n^- = N_2 \boxplus K_{n-2}$ and $D[K_n^-] = D[N_2] \boxplus D[K_{n-2}] = N_4 \boxplus H_{n-2}$, that is $D[K_n^-] = K_{4,2,......,2}$

4. Let G be a group and let Ω be a set of generators for G such that (i) if $x \in \Omega$ then $x^{-1} \in \Omega$, and (ii) $1 \notin \Omega$. The Cayley graph **Cay**(G, Ω) is the simple graph whose vertices are the elements of G and where x adj y if and only if $x^{-1}y \in \Omega$ (see [1]). Let now C_2 be a cyclic group of order 2. Then **Cay**($G \times C_2$, $\Omega \times C_2$) = D[**Cay**(G, Ω)].

A graph G is circulant when its adjacency matrix A is circulant, i.e. when every row distinct from the first one, is obtained from the preceding one by shifting every element one position to the right. Let $C(a_1 ..... a_n)$ be the circulant graph where $(a_1 .... a_n)$ is the first row of the adjacency matrix (for a suitable ordering of the vertices).

Proposition 11. A graph G is circulant if and only if D[G] is circulant. Specifically $D[C(a_1,..., a_n)] = C(a_1,......a_n, a_1,.....,a_n)$,

Let $D[G] = G \times K_2$ be the canonical double covering of G[20].

Proposition 12. D and R commutes, that is $D[R[G]] = R[D[G]]$ for every graph G.

Proof. The associativity and the commutativity of the direct product implies that

$$D[R[G]] = R[G] \times T_2 = G \times K_2 \times T_2 = G \times T_2 \times K_2 = D[G] \times K_2 = R[D[G]].$$

Let $[G] = G \boxtimes T_2$ be the strong double of G, and let $\overline{G}$ be the complement of G.

Proposition 13. For every graph G, $D[\overline{G}] = \overline{[G]}$.

Proof. If A is the adjacency matrix of G, the adjacency matrix of [G] is

$$A[G] = \begin{bmatrix} A & I + A \\ I + A & A \end{bmatrix}$$

Then

$$A[\overline{G}] = \begin{bmatrix} A(\overline{G}) & A(\overline{G}) \\ A(\overline{G}) & A(\overline{G}) \end{bmatrix} = \begin{bmatrix} J - I - A & J - I - A \\ J - I - A & J - I - A \end{bmatrix} = J - I - \begin{bmatrix} A & I + A \\ I + A & A \end{bmatrix}$$

that is $A([\overline{G}]) = J - I - A([G]) = A\overline{[G]}$. The proposition follows.

4. Spectral properties of double graphs

The eigenvalues, the characteristic polynomial and the spectrum of a graph are the eigenvalues, the characteristic polynomial and the spectrum of its adjacency matrix [5, p. 12].

Proposition 14. The characteristic polynomial of the double of a graph G on n vertices is

$$\varphi(D[G]; \lambda) = (2\lambda)^n \varphi(G; \lambda / 2)$$

In particular the spectrum of D[G] is {0..... 0, $2\lambda_1$,...... $2\lambda_n$} where λ_1, ...λ_n are the eigenvalues of G.

Proof. By Proposition 8 it follows that

$$\varphi(D[G]; \lambda) = \begin{vmatrix} \lambda I - A & -A \\ -A & \lambda I - A \end{vmatrix} = \begin{vmatrix} \lambda I - 2A & -A \\ \lambda I - 2A & \lambda I - A \end{vmatrix} = \begin{vmatrix} \lambda I - 2A & -A \\ 0 & \lambda I \end{vmatrix}$$

An integral graph is a graph all of whose eigenvalues are integers [5. p. 266].

Proposition 15. A graph G is integral if and only if $\mathcal{D}$[G] is an integral graph.

Proof. Since the characteristic polynomial of a graph is monic with integer coefficients its rational roots are necessarily integers. Then the claim immediately follows from Proposition 14.

Examples. 1. Since the spectrum of K_n is $(-1)^{n-1}(n-1)^1$ the spectrum of $H_n = \mathcal{D}[K_n]$ is $(-2)^{n-1} 0^n (2n-2)^1$ where –2 has multiplicity n–1 and 0 has multiplicity n.

2. The Petersen graph in an integral graph with spectrum $(-2)^4 1^5 3^1$. Then also its double is an integral graph whose spectrum is $(-4)^4 0^{10} 2^5 6^1$.

3. Since the characteristic polynomial of the path P_n is $\varphi(P_n; \lambda) = U_n(\lambda/2)$, where the $U_n(\lambda)$'s are the Chebyshev polynomials of the second kind, it follows that the characteristic polynomial of $\mathcal{D}[P_n]$ is $\varphi(\mathcal{D}[P_n]; \lambda) = (2\lambda)^n U_n(\lambda/4)$.

4. Let $c_3(G)$ be the number of triangles (i.e. 3-cycles) of G. If A is the adjacency matrix of G then $c_3(G) = \frac{1}{6} tr(A^3) = \frac{1}{6}(\lambda_1^3 + \cdots + \lambda_n^3)$ where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A. Then Proposition 14 implies that $c_3(\mathcal{D}[G]) = 8.C_3(G)$.

Two graphs are cospectral when they are non-isomorphic and have the same spectrum [1, p. 12; 5, p. 156]. From Proposition 14 and Theorem 31 we have the following property.

Proposition 16. Two graphs G_1 and G_2 are cospectral if and only if their doubles $\mathcal{D}[G_1]$ and $\mathcal{D}[G_2]$ are cospectral.

Given two cospectral graphs G_1 and G_2, it is always possible to construct an infinite sequence of cospectral graphs. Indeed $\mathcal{D}^k[G_1]$ and $[G_2]$ are cospectral for every $k \in \mathbb{N}$.

5. Strongly regular graphs

A graph G is k-regular if every vertex has degree k.

Proposition 17. A graph G is k-regular if and only if $\mathcal{D}$[G] is $2k$-regular.

A simple graph G is strongly regular with parameters (n, k, λ, μ) when it has n vertices, is k-regular, every adjacent pair of vertices has λ common neighbors and every non-adjacent pair has μ common neighbors. For instance the complete graph K_n is (n,n–1, n–2, 0)-strongly regular, the complete bipartite graph $K_{n,n}$ is (2n, n, 0, n)-strongly regular and the hyperoctahedral graph H_n is (2n, 2n–2,2n–4, 2n–2)-strongly regular.

Connected strongly regular graphs, distinct from the complete graph, are characterized [5, p. 103] as the connected regular graphs with exactly three distinct eigenvalues. Hence if G is strongly regular its double is not necessarily strongly regular. For instance the Petersen graph is a (10, 3, 0, 1)-strongly regular graph with the three distinct eigenvalues –2, 1. 3, but its double is not strongly regular having the four distinct eigenvalues –4, 0, 2, 6. Strongly regular double graphs can however be completely characterized, as we will do in Proposition 19. To give such a characterization we need the following properties.

Strongly regular graphs with one zero eigenvalue are characterized as follows [5. p. 163]: a regular graph G has eigenvalues k, 0, λ_3 if and only if the complement of G is the sum of $1-k/\lambda_3$ complete graphs of order $-\lambda_3$. Equivalently, a regular graph has three distinct eigenvalues of which one is zero if and only if it is a multipartite graph $K_{m(n)}$.

The only disconnected strongly regular graphs are finite sums of complete graphs of the same order [4].

Lemma 18. A complete multipartite graph $K_{m(n)}$ is a double graph if and only if n is even. In particular, the complete graph K_n is never a double graph.

We can now characterize the strongly regular double graphs.

Proposition 19. For any graph G the following characterizations hold:

1. D[G] is a connected strongly regular graph if and only if G is a complete multipartite graph $K_{m(n)}$
2. D[G] is a disconnected strongly regular graph if and only if G is a completely disconnected graph N_n.

Proof. 1. If G = $K_{m(n)}$ then D[G]=$K_{m(2n)}$. Conversely, suppose that D[G] is connected and strongly regular. Since D[G] cannot be a complete graph, it has 3 distinct eigenvalues, one of which is zero. Then it is a complete multipartite graph $K_{m(2n)}$ and consequently G is the complete multipartite graph $K_{m(n)}$.

2. If D[G] is a disconnected strongly regular graph then it is a sum of complete graphs of the same order. Since the complete graph is never a double graph, the only possibility is that D[G] = N_{2_n} and hence G = N_n.

In general double graphs are not characterized by their spectrum. However, since this is true for complete bipartite graphs, we have that:

Proposition 20. Strongly regular double graphs are characterized by their spectrum.

6. Complexity and Laplacian spectrum

Let t (G) be the complexity of the graph G. i.e. the number of its spanning trees. It is well known [3] that

$$t(G) = \frac{1}{n^2}\det(L + J), \qquad ...(1)$$

where n is the number of vertices of G, L is the Laplacian matrix of G and J, as before, is the n×n matrix all of whose entries are equal to 1.

Theorem 21. The complexity of the double of a graph G on n vertices with degrees $d_1, d_2, ...d_n$ is

$$t(D[G] = 4^{n-1} d_1 d_2 .....d_n t(G). \qquad ...(2)$$

Proof. Let $v_1....v_n$ be the vertices of G and $d_1, ... d_n$ their degrees. As known the Laplacian matrix L of G is equal to D–A where D is the diagonal matrix diag($d_1,...., d_n$) and A is the adjacency matrix of G. Then the Laplacian matrix of D[G] is

$$D[L] = D[D] - D[A] = \begin{bmatrix} 2D & O \\ O & 2D \end{bmatrix} - \begin{bmatrix} A & A \\ A & A \end{bmatrix} = \begin{bmatrix} 2D - A & -A \\ -A & 2D - A \end{bmatrix} \qquad ...(3)$$

Hence it follows that

$$t(D[G]) = \frac{1}{(2n)^2}\begin{vmatrix} 2D - A + J & -A + J \\ -A + J & 2D - A + J \end{vmatrix}$$

Subtracting the first row to the second row and then adding the second column to the first one, we have

$$t(D[G]) = \frac{1}{4n^2}\begin{vmatrix} 2D - A + J & -A + J \\ 2D & 2D \end{vmatrix} = \frac{1}{4n^2}\begin{vmatrix} 2D - 2A + 2J & -A + J \\ O & 2D \end{vmatrix}$$

Then

$$t(D[G]) = \frac{1}{4n^2}|2D - 2A + 2J|.|2D| = \frac{4^n}{4n^2}|L + J|.|D| = 4^{n-1}|D \backslash t(G)$$

and the theorem follows.

As an immediate consequence we have the following:

Theorem 22. The complexity of the double of a k-regular graph G on n vertices is

$$t(D[G]) = 4^{n-1} k^n t(G). \qquad ...(4)$$

Examples. 1. The double of the complete graph K_n is the hyperoctahedral graph H_n. Since K_n is (n–1)-regular, (4) implies that $t(H_n) = 4^{n-1} (n-1)^n t(K_n) = 4^{n-1}(n-1)^n n^{n-2}$.

2. Let P_n be the path on a vertices, with $n \geq 2$. Then $t(D[P_n]) = 4^{n-1}2^{n-2} t(P_n) = 2^{3n-4}$

3. Let C_n be the cycle on n vertices, with $n \geq 3$. Then $t(D[C_n]) = 4^{n-1}2^n t(C_n) = 2^{3n-2}n$.

4. Let $F_n = K_1 \sqcup P_n$ be a 4 *fan*, with n=2. Then $t(D[P_n]) = 4^{n+1} 3^{n-2} n\, t(F_n)$. Sincer $t(F_n) = f_{2n}$, where the f_n's are the Fibonacci numbers [7], it follows that $t(D[F_n]) = 4^{n+1} 3^{n-2} n f_{2n}$.

Since any tree has only one spanning tree, the second example can be generalized as follows:

Theorem 23. Let T be a tree on n vertices with degrees $d_1, \ldots, d_n$. Then

$$t(D[T]) = 4^{n-1} d_1 \ldots d_n. \qquad ...(5)$$

It follows that the complexity of the double of a tree depends only on the degrees of the vertices of the tree itself. For instance, the graphs $D[T_1]$ and $D[T_2]$ in Fig. 3 have the same number t =73728 of spanning trees, because they are the double of two trees T_1 and T_2 on seven vertices with the same distribution of degrees (3, 3, 2, 1, 1, 1, 1). Since T_1 and T_2 are not isomorphic, $[T_1]$ and $[T_2]$ are non-isomorphic graphs too (by Theorem 31, as we shall see in Section 8). Finally, using identity (3), the following proposition can be proved.

Proposition 24. Let G be a graph on n vertices with degrees $d_1, d_2, \ldots, d_n$ and let $\{\lambda_1, \ldots, \lambda_n\}$ be its Laplacian spectrum. Then the Laplacian spectrum of $D[G]$ is $\{2d_1, \ldots, 2d_n, 2\lambda_1, \ldots, 2\lambda_n\}$. In particular, G has an integral Laplacian spectrum if and only if the same holds for [G].

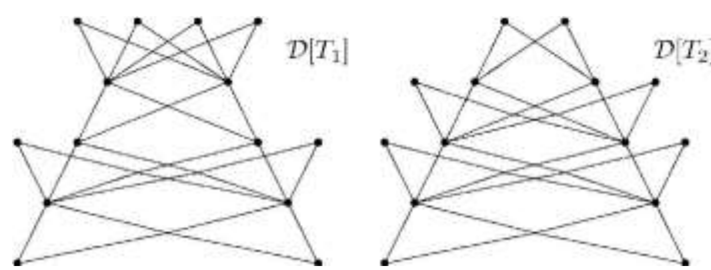

Fig. 3. Non-isomorphic graphs with the same complexity t=73 728.

7. Independent sets

An independent set of vertices of a graph G is a set of vertices in which no pair of vertices is adjacent. Let $k[G]$ be the set of all independent subsets of size k of G and let $i_k(G)$ be its size. The independence polynomial of G is defined as

$$I(G;x) = \sum_{k \geq 0} \sum_{s \in k[G]} x^{|S|} = \sum_{k \geq 0} i_k(G) x^k.$$

Proposition 25. For any graph G we have $k[D[G]] \cdot k[G] \times 2^k$, where 2 = {0, 1}. In particular $i_k(D[G]) = 2^k i_k(G)$ and $I(D[G]; x) = I(G, 2x)$.

Proof. Let the vertices of G be linearly ordered in some way. Let S = {(v_1, w_1)..... (v_k, w_k)) be an independent set of D[G] G × T_2. Since T_2 is a total graph, it follows that π_1 (S) = {v_1,......v_k) is an arbitrary independent subset of G and π_2(S) is equivalent to an arbitrary binary sequence (w_1,..... w_k) of length k (where the order is established by the order of π_1 (S) induced by the order of V (G)). The claim follows.

The (vertex) independence number α(G) of a graph G is the maximum size of the independent sets of vertices of G. Equivalently, α(G) is the degree of the polynomial I (G, x). Then Proposition 25 implies the following:

Proposition 26. For any graph G we have that α(D[G]) = 2α(G).

8. Morphisms

A morphism $f : G \to H$ between two graphs G and H is a function from the vertices of G to the vertices of H which preserves adjacency (i.e. v adj w implies $f(v)$ adj $f(w)$, for every v, $w \in$V(G)) [8,10]. An isomorphism between two graphs is an invertible morphism.

Let **Hom**(G, H) be the set of all morphisms from G to H and let $2^{V[G]}$ be the set of all functions from V(G) to 2 = {0, 1}.

Lemma 27. For every graph G and H. **Hom**(G. D[H]) = **Hom**(G, H) × $2^{V[G]}$.

Proof. From the universal property of the direct product (in the categorical sense [2]) we have **Hom**(G, G_1 × G_2) = **Hom**(G, G_1) × **Hom**(G, G_2). Since D[G]=G × T_2 and **Hom**(G, T_2) = $2^{V[G]}$, the lemma follows.

A k-walk, or a walk with k steps, in a graph G is a sequence v_1, v_2 .... v_k of vertices of G such that v_i adj v_i +1 for i = 1.....k–1. A k-walk is closed when v_k adj v_1. Then a k-walk is a morphismy y: $p_k \to G$ while a closed k-walk is a morphism $\gamma : C_k \to G$.. Let w_k(G) and $\overline{w_k}$ (G) be the number of all k-walks and closed k-walks of G, respectively. Hence w_k (G) = |**Hom**(P_k, G) and $\overline{w_k}$ (G) = |**Hom**(C_k, G). Lemma 27 immediately implies the following:

Proposition 28. For any graph G the number of k-walks and closed k-walks on D[G] are w_k(D[G]) = $2^k w_k$ (G) and $\overline{w_k}$ (D[G])=$2^k \overline{w_k}$ (G).

To prove Theorem 31 we recall the following theorems:

Theorem 29 (Lovász [15.16]). Two graphs G_1 and G_2 are isomorphic if and only if for every graph G the number of morphisms from G to G_1 is equal to the number of morphisms from G to G_2.

Theorem 30 (Imrich and Klavžar [13, p. 190]). If $G \circ H$ - $G' \circ H'$ and |V (H) | = |V(H')| then $H \cdot H'$ and $G \cdot G'$.

We can now prove the following:

Theorem 31. Two graphs G_1 and G_2 are isomorphic if and only if their doubles D [G_1] and [G_2] are isomorphic.

Proof. The claim is an immediate consequence of Theorem 30. However, it is interesting to observe that it is also a consequence of Lovasz's theorem [10]. Indeed, if two graphs are isomorphic it is clear that their doubles are isomorphic too. Conversely, if D[G_1] and [G_2] are isomorphic then |**Hom**(G, [G_1])| = |**Hom**(G,D[G_2])| for every graph G. From Lemma 27 it follows that |**Hom**(G, G_1).$2^{|V[G]|}$=|**Hom**(G, G_2)|.$2^{|V[G]}$, that is |**Hom**(G, G_1)| = |**Hom**(G, G_2), for every graph G. Hence, by Lovasz's theorem, G_1 and G_2 are isomorphic.

We now extend D to morphisms in the following way: for any graph morphism $f : G \to H$ let $D[f] : D[G] \to D[H]$ be the morphism defined by $D[f](v, k) = (f(v), k)$ for every $(v, k) \in D[G]$. In this way is an endofunctor of the category of finite simple graphs and graph morphisms.

A morphism $r : G \to H$ between two graphs G and H is a retraction if there exists a morphism s: H $\to$ G such that $sr = r \circ s = 1_H$. If there exists a retraction $r : G \to H$ then H is a retract of G. Since D is a functor it preserves retractions and retracts.

Proposition 32. Every graph G is a retract of its double. More generally every retract of G is also a retract of D[G].

Proof. Consider the morphisms $r : D[G] \to G$ and $s : G \to D[G]$ defined by $r(v, k) = v$ for every $(v, k) \in V(D[G])$ and $s(v) = (v, 0)$ for every $v \in V(G)$. Then r, which is the projection of $G \times T_2$ on G, is a retraction. The second part of the proposition follows from the fact that D is a functor and the composition of retractions is a retraction.

Let π be a partition in independent classes of G. Then the quotient G/π is the graph whose vertices are the classes of π and X adj Y when there exist two vertices $v \in X$ and $w \in Y$ adjacent in G. The kernel of a graph morphism f: G $\to$ H is the partition induced from f on the vertices of G. Clearly the kernel of a graph morphism is a partition in independent blocks.

Proposition 33. For every graph G let $r : D[G] \to G$ be the projection on G and let be its kernel. Then $D[G]/\pi = G$.

Proposition 34. For every graph G_1 and G_2, $G_1 \times G_2$ is a retract of $D[G_1] \times [G_2]$.

Proof. From Proposition 7 it follows that $D[G_1] \times [G_2] = D^2[G \times G_2]$. Then the claim is implied by Proposition 32.

A (proper) coloring of a graph G is a morphism c : G $\to$ K. The chromatic number χ(G) of a graph G is the minimum number of colors needed to color the vertices of G. If there exists a morphism f: G $\to$ H then every coloring c : H $\to$ K_n of H can be lifted to a coloring of G by the composition $G \xrightarrow{f} H \xrightarrow{c} K_n$. Hence it follows that $\chi(G) \leq \chi(H)$. In particular χ(G) = χ(H) whenever I is a retract of G. Then it follows:

Proposition 35. For any graph G. $\chi(D[G]) = \chi(G)$. More generally, $\chi(D[G]) = \chi(H)$ whenever H is a retract of G.

The chromatic polynomial $\chi(G; x)$ of a graph G is defined as the polynomial that evaluated in any natural number m gives the numbers of proper colorings of G with m colors, that is $\chi(G; m)$ = |**Hom**(G. K_m)|.

We now define a hyperoctahedral coloring of G as any morphism $\gamma : G \to H_m$ from G to any hyperoctahedral graph. Then the number of hyperoctahedral colorings can be expressed in terms of the number of ordinary colorings. Specifically:

Proposition 36. For any graph G on n vertices. |**Hom**(G, H_m) = 2n . $\chi(G; m)$.

Proof. Lemma 27 and $H_m = D[K_m]$ imply that **Hom**(G, H_m) = **Hom**(G, K_m) × $2^{V(G)}$,

The clique number x(G) is the size of a maximal clique contained in G. Equivalently x(G) is the maximal k such that **Hom**(K_k, G) $\neq \phi$. It follows that x(G) = x(H) whenever H is a retract of G. Hence:

Proposition 37. For any graph G, x(D[G]) = x(G). More generally, x(D[G])=x(H) whenever H is a retract of G.

Let now $x_k(G)$ be the number of all cliques of order k contained in G. Since every morphism from K_k to G is necessarily injective, it follows that $x_k(G) = (1/k!)|\mathbf{Hom}(K_k,G)|$. Then Lemma 27 implies the following:

Proposition 38. For every graph G. $x_k(\mathcal{D}[G]) = 2^k x_k(G)$.

A graph G is a core when no proper subgraph of G is a retract of G. A retract of G is a core of G if it is a core. Every finite graph has a core and it is unique up to isomorphisms [8, p. 114]. A double graph is never a core but we have the following:

Proposition 39. If H is the core of G then it is also the core of $\mathcal{D}$[G]. In particular, if G is a core then it is the core of [G].

Proof. Being a retract of G, H is also a retract of $\mathcal{D}$[G]. Since H is a core and every graph has just one core, up to isomorphisms, it follows that H is the core of $\mathcal{D}$[G].

A median of three vertices of a connected graph is a vertex that lies simultaneously on geodesics between any two of them. A graph G is a median graph when every triple of (not necessarily distinct) vertices of G has a unique median [8,13]. Median graphs are characterized as retracts of hypercubes [13, p. 76].

Proposition 40. If $\mathcal{D}$[G] is a median graph then also G is median.

Proof. If $\mathcal{D}$[G] is a median graph then it is a retract of a hypercube. Since G is a retract of $\mathcal{D}$[G], by Theorem 32, it follows that it is also a retract of a hypercube.

In general, however, if G is median it does not necessarily follow that $\mathcal{D}$[G] is median. For instance the star $K_{1,3}$ is median (being a tree) while its double $K_{2,6}$ is not median (fails the uniqueness of median vertices).

A morphism $f: G \to H$ is full when v adj w if and only if f(v) adj f(w), for every v, $w \in$ V (G). We have the following characterization theorem:

Theorem 41. H is a double graph if and only if there exists a partition π of H in independent classes each of size 2 such that the canonical projection $p: H \to H/\pi$ is a full morphism.

Proof. It immediately follows from the identity $\mathcal{D}$[G] G oN_2.

Equivalently we have the following characterization theorem:

Theorem 42. For every graph G and H. G = $\mathcal{D}$[H] if and only if there exists a function $f: G \to H$ such that (i) f preserves and reflects adjacency (i.e. v_1 adj v_2 in G if and only if f (v_1) adj f (v_2) in H. for every v_1, $v_2 \in$ G), (ii) f is 2-regular (i.e. every vertex of H has exactly two preimages).

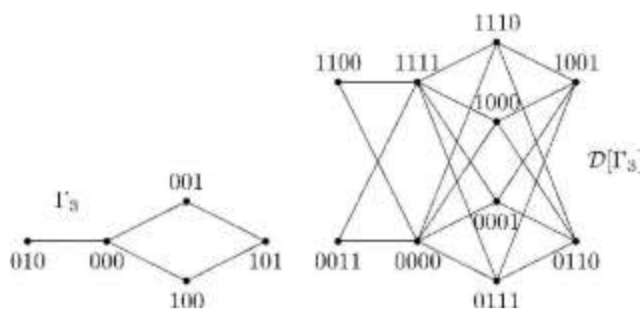

Fig. 4. A Fibonacci cube and its double.

9. An example: the double of generalized Fibonacci cubes

In this section we will consider a generalization of the motivating example mentioned in the Introduction. Let L be a (finite) set of words of a given length over some alphabet. The Hamming graph generated by L is the graph with vertex set L where two vertices are adjacent if and only if they have Hamming distance equal to 1, that is if and only if they differ exactly in one position.

A k-Fibonacci string is a binary string without k consecutive ones. Let $F_n^{[k]}$ be the set of all k-Fibonacci strings with length n. The generalized Fibonacci cube $\Gamma_n^{[k]}$ is the Hamming graph generated by the $F_n^{[k]}$. In particular for $k = 2$ we have the ordinary Fibonacci cube Γ_n (Fig. 4).

The binary strings 1010..... and 0101...... with k letters will be called zigzags of length k. or simply k-zigzags. Let $W_n^{[k]}$ be the set of all binary strings of length n without k-zigzags (as factors).

Let B_n be the set of all binary strings of length n. Let $\xi : B_{n-1} \to B_n$ be the function defined by $\xi(a_1a_2.......a_na_{n-1}) = b_1b_2.....b_n$, where $b_i = xor(a_i, a_{i-1})$ for i = 1, 2,..., n. where xor(0, 0) = xor(1, 1)=0 and xor(0, 1) = xor(1, 0) = 1. Consider now the restriction φ of ξ to $W_{n-1}^{[k+1]}$. Since ξ(1010.....) =111.... and ξ(0101....)=111....... it follows that the image of φ is the set of all binary strings of length n without k consecutive 1's, that is the function $\varphi : W_{n-1}^{[k+1]} \to F_n^{[k+1]}$ is well defined. This function is surjective and any element of the codomain has exactly two preimages. Now, instead of considering the Hamming graph generated by $W_{n-1}^{[k-1]}$, we consider the graph $W_{n-1}^{[k-1]}$ obtained by endowing $W_{n-1}^{[k-1]}$ with the graph structure induced by $\Gamma_n^{[k]}$ in order that φ becomes a graph morphism between $W_{n-1}^{[k-1]}$ setting w_1 adj w_2 if and only if φ (w_1) adj (w_2) in we define the adjacency on $W_{n-1}^{[k-1]}$ By Theorem 42, it immediately follows that $W_{n-1}^{[k-1]} = D[\Gamma_n^{[k]}]$.

10. Chromatic index

The chromatic index χ' (G) of a graph G is the minimum number of colors needed to color the edges of G so that adjacent edges are colored differently. By Vizing's theorem the chromatic index of a graph G with maximum degree $\Delta = \Delta(G)$ is equal to Δ (class 1 graphs) or Δ +1 (class 2 graphs) [6,11].

Since every bipartite graph is of class 1 (König's theorem, [6, p. 25]), it follows that the double of a bipartite graph is of class 1. This result can be generalized as follows.

Theorem 43. If G is of class 1 then also D[G] is of class 1.

Proof. Let c be a proper edge coloring of G using all the colors in a set C of size $= \Delta$ (G). The coloring c can be represented by the matrix A_c obtained from A by replacing every element a_{ij} =1 with the color c(i, j) assigned to the edge v_iv_j by c. Let C' be a new set of Δ colors such that $C \cap C' = \phi$ and let $\phi : C \to C'$ be a bijection. We have a new coloring c' of the edges of G by assigning to the edge v_iv_j the color c'(i, j) = $\phi(c(i, j))$. Let $A_{c'}$ be the matrix Ac A representing c'. Then the matrix $\begin{bmatrix} A_c & A_{c'} \\ A_{c'} & A_c \end{bmatrix}$ represents a proper coloring of the edges of D[G] where exactly 2 Δ colors are used.

What can be said if G is a class 2 graph? The double is not necessarily of class 2. For instance the complete graph K_n is of class 2 for $n \geq 3$ odd but its double H_n is of class 1 [6. p. 28]. More generally, the complete h-partite graph $K_{h(k)}$ is of class 2 if both h and k are odd and it is of class 1 otherwise [14, 6, p. 28]. Then for h and k odd $K_{h(k)}$ is of class 2 but its double $K_{h(2k)}$ is of class 1. Similarly the cycle C_n is of class 2 when n is odd, but its double $K_{h(2k)}$ is of class 1 [19, 6, p. 28). All the eight connected graphs of class 2 with at most 6 vertices [6. p. 37] and the Petersen graph have a double of class 1. All the graphs of class 2 we considered have a double of class 1. This suggests the possibility that all double graphs are of class 1.

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